LAMPIRAN

LAMPIRAN 1. Pembuktian Proporsi

Pembuktian Proporsi (39)

TC (Q,S,S_{maks}) =

$$ABDQ + BCD + \frac{DK}{Q} + \frac{DE}{Q} + \frac{H\left[Q(1-D/P)-S_{maks}\right]^2}{2Q(1-D/P)} + \frac{\pi S^2{}_{maks}}{2Q(1-D/P)} + \frac{\pi S_{maks} D}{Q}$$

Jika urutan (*sequence*) produksi tetap, TC hanya berupa fungsi Q. QSOL dilambangkan sebagai Q solusi awal

$$ABD + H\left(\frac{1-AD}{2}\right) - Q^{-2}D(K+E+\pi Smaks) = \frac{S^2 maks(2-2AD)}{4Q^2 - 8Q^2AD + 4QCD + 4Q^2A^2D^2}(H+\pi)$$

Pembuktian Proporsi (40)

$$\frac{r_i h_i}{(s_i+Qr_it_i)} > \frac{r_j h_j}{(s_j+Qr_jt_j)}$$

Flow time dari komponen i dan j dalam S adalah T-Ta dan T-Ta-Qr_it_i- s_i ,

sedangkan untuk biaya WIP:

T

	s_i	Qr_it_i	s_j	Qr_jt_j	

S Ta Tb T

T

	s_j	Qr_jt_j	s_i	Qr_it_i	

S' Ta Tb T

WIP(S) = Qr_ih_i(T-Ta)+ Qr_jh_j(T-Ta-Qr_jt_j- s_j) = $Qr_ih_i(Qr_it_i+ s_j+Qr_jt_j)$+ $Qr_jh_j(Qr_jt_j)$

WIP(S')= Qr_jh_j(T-Ta)+ Qr_ih_i(T-Ta-Qr_it_i- s_i) = $Qr_jh_j(Qr_jt_j+ s_i+Qr_it_i)$+ $Qr_ih_i(Qr_it_i)$

WIP(S)-WIP(S') = $Qr_ih_i(Qr_it_i+ s_j+Qr_jt_j)$+ $Qr_jh_j(Qr_jt_j)$ - $Qr_jh_j(Qr_jt_j+ s_i+Qr_it_i)$+ $Qr_ih_i(Qr_it_i)$

Pembuktian Proporsi (41)

TC (Q,S,S_{maks}) =

$$QBDt + D\sum_{i=1}^{N}\sum_{k=1}^{N} h_{(i)} r_{(i)} \left(S_{(k)} + Qr_{(k)} t_{(k)}\right) + \frac{DK}{Q} + \frac{DE}{Q} + \frac{H\left[Q(1-D/P)-S_{maks}\right]^2}{2Q(1-D/P)}$$

$$+\frac{\hat{\pi} S^2{}_{maks}}{2Q(1-D/P)} + \frac{\pi S_{maks} D}{Q}$$

Jika urutan (sequence) produksi tetap, TC hanya berupa fungsi Q. Q* dilambangkan sebagai Q optimum.

$$D\left(BDt + \sum_{i=1}^{N}\sum_{k=1}^{N} h_{(i)} r_{(i)} r_{(k)} t_{(k)}\right) + H\left(\frac{1-AD}{2}\right) - Q^{-2}(K + E + \pi S_{maks})$$

$$= \frac{S^2{}_{maks}(2-2AD)}{4Q^2 - 8Q^2AD + 4QCD + 4Q^2A^2D^2 - 4QACD^2 + C^2D^2}(H+\pi)$$

Jika Ukuran *batch* tetap, TC hanya berupa fungsi S_{maks}

$$\frac{2S_{maks}(2Q - 2ADQ + 2CD)}{4Q^2 - 8Q^2AD + 4QCD + 4Q^2A^2D^2 - 4QACD^2 + C^2D^2}(H+\pi) = \left(H - \frac{\pi D}{Q}\right)$$

Pembuktian Proporsi (42)

Penjelasan proporsi (42) sama dengan proporsi (40)

$$\frac{r_i h_i}{(s_i + Qr_i t_i)} > \frac{r_j h_j}{(s_j + Qr_j t_j)}$$

Flow time dari komponen i dan j dalam S adalah T-Ta dan T-Ta-Qr_it_i- s_i ,

sedangkan untuk biaya WIP:

T

	s_i	Qr_it_i	s_j	Qr_jt_j	

S Ta Tb T

T

	s_j	Qr_jt_j	s_i	Qr_it_i	

S' Ta Tb T

$WIP(S) = Qr_i h_i(T\text{-}Ta) + Qr_j h_j(T\text{-}Ta\text{-}Qr_j t_j\text{-}s_j) = Qr_i h_i(Qr_i t_i + s_j + Qr_j t_j) + Qr_j h_j(Qr_j t_j)$

$WIP(S') = Qr_j h_j(T\text{-}Ta) + Qr_i h_i(T\text{-}Ta\text{-}Qr_i t_i\text{-}s_i) = Qr_j h_j(Qr_j t_j + s_i + Qr_i t_i) + Qr_i h_i(Qr_i t_i)$

$WIP(S)\text{-}WIP(S') = Qr_i h_i(Qr_i t_i + s_j + Qr_j t_j) + Qr_j h_j(Qr_j t_j) - Qr_j h_j(Qr_j t_j + s_i + Qr_i t_i) +$
$Qr_i h_i(Qr_i t_i)$

Pembuktian Proporsi (43)

Penjelasan Proporsi (43) sama dengan proporsi (41)

$$TC(Q,S,S_{maks},K) =$$

$$QBDt + D\sum_{i=1}^{N}\sum_{k=1}^{N} h_{(i)} r_{(i)}\left(S_{(k)} + Qr_{(k)}t_{(k)}\right) + \frac{DK}{Q} + \frac{DE}{Q} + \frac{H\left[Q\left(1-\frac{D}{P}\right) - S_{maks}\right]^2}{2Q\left(1-\frac{D}{P}\right)}$$

$$+ \frac{\hat{\pi} S^2_{maks}}{2Q\left(1-\frac{D}{P}\right)} + \frac{\pi S_{maks} D}{Q} + i\left(a - b\ln(K)\right)$$

Misalkan S tetap, S_{maks} tetap, dan K tetap maka TC (Q ,S, S_{maks}, K) hanyalah merupakan fungsi Q.

Q^* menunjukkan ukuran *batch* optimum; yang meminimasi TC(Q, S, S^*_{maks}, K)

$$D\left(BDt + \sum_{i=1}^{N}\sum_{k=1}^{N} h_{(i)} r_{(i)} r_{(k)} t_{(k)}\right) + H\left(\frac{1-AD}{2}\right) - Q^{-2}(K + E + \pi S_{maks})$$

$$= \frac{S^2_{maks}(2-2AD)}{4Q^2 - 8Q^2AD + 4QCD + 4Q^2A^2D^2 - 4QACD^2 + C^2D^2}(H+\pi)$$

Pembuktian Proporsi (44)

Penjelasan Proporsi (44) sama dengan proporsi (41)

$$TC(Q,S,S_{maks},K) =$$

$$QBDt + D\sum_{i=1}^{N}\sum_{k=1}^{N} h_{(i)}r_{(i)}\left(S_{(k)} + Qr_{(k)}t_{(k)}\right) + \frac{DK}{Q} + \frac{DE}{Q} + \frac{H\left[Q\left(1-\frac{D}{P}\right) - S_{maks}\right]^2}{2Q\left(1-\frac{D}{P}\right)}$$

$$+ \frac{\hat{\pi}S^2_{maks}}{2Q\left(1-\frac{D}{P}\right)} + \frac{\pi S_{maks}D}{Q} + i\left(a - b\ln(K)\right)$$

Misalkan Q tetap, S tetap, dan K tetap maka TC (Q ,S, S_{maks}, K) hanyalah merupakan fungsi S_{maks}. S^*_{maks} menunjukkan ukuran batas optimum ***shortage*** yang diijinkan yang meminimasi TC(Q, S, S^*_{maks}, K)

$$\frac{2S_{maks}(2Q - 2ADQ + 2CD)}{4Q^2 - 8Q^2AD + 4QCD + 4Q^2A^2D^2 - 4QACD^2 + C^2D^2}(H+\pi) = \left(H - \frac{\pi D}{Q}\right)$$

Pembuktian Proporsi (45)

$$TC(Q,S,S_{maks},K) =$$

$$QBDt + D\sum_{i=1}^{N}\sum_{k=1}^{N} h_{(i)}r_{(i)}\left(S_{(k)} + Qr_{(k)}t_{(k)}\right) + \frac{DK}{Q} + \frac{DE}{Q} + \frac{H\left[Q\left(1-\frac{D}{P}\right) - S_{maks}\right]^2}{2Q\left(1-\frac{D}{P}\right)}$$

$$+ \frac{\hat{\pi}S^2_{maks}}{2Q\left(1-\frac{D}{P}\right)} + \frac{\pi S_{maks}D}{Q} + i\left(a - b\ln(K)\right)$$

Q tetap, S tetap, dan S_{maks} tetap maka TC (Q ,S, S_{maks}, K) hanyalah fungsi K. K* menunjukkan ukuran biaya ***setup*** optimum; yang meminimasi TC(Q, S, S^*_{maks}, K)

$$\frac{D}{Q} = \frac{ib}{K}$$

LAMPIRAN 2. Contoh Perhitungan Penelitian Tahap I

<u>Tahap I - D1000</u>

I	ri	ki	hi	si	ti	rihi	riti
1	1	80	0.22	1.25	0.5	0.22	0.500
2	2	80	0.06	1.25	0.2	0.12	0.400
3	2	40	0.03	2	0.05	0.06	0.100
4	4	40	0.01	1.5	0.03	0.04	0.120
5	4	20	0.1	0.8	0.1	0.4	0.400
6	4	20	0.02	0.5	0.08	0.08	0.320
7	6	10	0.07	2	0.04	0.42	0.240
8	8	10	0.005	0.2	0.01	0.04	0.080
9	8	10	0.04	1.2	0.02	0.32	0.160
10	10	10	0.015	1	0.06	0.15	0.600
		320		11.7		1.85	2.92

x = 0.1

N =	t =	TPC =	E =	D =	H =	π	$\hat{\pi}$	i	θ
10	10	1	250	1000	3	1	5	0.15	200
A =	B =	C =	K =					a	b
12.92	1.85	11.7	320					10949.7	1898.24

QLB =	-0.90564285									
QSOL =	271	271.41								
Qmin =	**271**									
	1	2	3	4	5	6	7	8	9	10
hiri	0.22	0.12	0.06	0.04	0.4	0.08	0.42	0.04	0.32	0.15
riti	0.500	0.400	0.100	0.120	0.400	0.320	0.240	0.080	0.160	0.600
si	1.25	1.25	2	1.5	0.8	0.5	2	0.2	1.2	1
	0.001663516	0.00113	0.00213	0.0012	0.00379	0.00095	0.00647	0.00189	0.00742	0.00095
	(10.6.2.4.1.8.3.6.7.9)									
$S_{solmaks}$ =	301982.5	=	**301983**							

	10	6	2	4	1	8	3	5	7	9
hiri	0.15	0.08	0.12	0.04	0.22	0.04	0.06	0.4	0.42	0.32
si	1	0.5	1.25	1.5	1.25	0.2	2	0.8	2	1.2
riti	0.6	0.32	0.4	0.12	0.5	0.08	0.1	0.4	0.24	0.16
Qriti	157.2	83.84	104.8	31.44	131	20.96	26.2	104.8	62.88	41.92
TC =	**5560466.2**									

Tahap II

I	ri	ki	hi	si	ti	rihi	riti
1	1	80	0.22	1.25	0.5	0.22	0.500
2	2	80	0.06	1.25	0.2	0.12	0.400
3	2	40	0.03	2	0.05	0.06	0.100
4	4	40	0.01	1.5	0.03	0.04	0.120
5	4	20	0.1	0.8	0.1	0.4	0.400
6	4	20	0.02	0.5	0.08	0.08	0.320
7	6	10	0.07	2	0.04	0.42	0.240
8	8	10	0.005	0.2	0.01	0.04	0.080
9	8	10	0.04	1.2	0.02	0.32	0.160
10	10	10	0.015	1	0.06	0.15	0.600
		320		11.7		1.85	2.92

x = 0.1

N =	t =	TPC =	E =	D =	H =	π	$\hat{\pi}$	i	θ
10	10	1	250	1000	3	1	5	0.15	200
A =	B =	C =	K =					a	b
12.92	1.85	11.7	320					10949.7	1898.24

Q = **314** 310.74

	10	6	2	4	1	8	3	5	7	9
hiri	0.15	0.08	0.12	0.04	0.22	0.04	0.06	0.4	0.42	0.32
riti	0.6	0.32	0.4	0.12	0.5	0.08	0.1	0.4	0.24	0.16
si	1	0.5	1.25	1.5	1.25	0.2	2	0.8	2	1.2
	0.000948167	0.00095	0.001132	0.0012	0.00166	0.00189	0.00213	0.00379	0.00647	0.00742

(10.6.2.4.1.8.3.5.7.9)

$S_{solmaks}$ = 93665.95 = **93662.95**

	10	6	2	4	1	8	3	5	7	9
hiri	0.15	0.08	0.12	0.04	0.22	0.04	0.06	0.4	0.42	0.32
si	1	0.5	1.25	1.5	1.25	0.2	2	0.8	2	1.2
riti	0.6	0.32	0.4	0.12	0.5	0.08	0.1	0.4	0.24	0.16
Qriti	157.2	83.84	104.8	31.44	131	20.96	26.2	104.8	62.88	41.92

TC = **259649.4693**

tahap III

I	ri	ki	hi	si	ti	rihi	riti
1	1	80	0.22	1.25	0.5	0.22	0.500
2	2	80	0.06	1.25	0.2	0.12	0.400
3	2	40	0.03	2	0.05	0.06	0.100
4	4	40	0.01	1.5	0.03	0.04	0.120
5	4	20	0.1	0.8	0.1	0.4	0.400
6	4	20	0.02	0.5	0.08	0.08	0.320
7	6	10	0.07	2	0.04	0.42	0.240
8	8	10	0.005	0.2	0.01	0.04	0.080
9	8	10	0.04	1.2	0.02	0.32	0.160
10	10	10	0.015	1	0.06	0.15	0.600
		320		11.7		1.85	2.92

x = 0.1

N =	t =	TPC =	E =	D =	H =	π	$\hat{\pi}$	i	θ
10	10	0.01	250	1000	3	1	5	0.15	200
A =	B =	C =	K =					a	b
12.92	1.85	11.7	320					10949.7	1898.24
Q =	**314**	310.174							

	10	6	2	4	1	8	3	5	7	9
hiri	0.15	0.08	0.12	0.04	0.22	0.04	0.06	0.4	0.42	0.32
riti	0.6	0.32	0.4	0.12	0.5	0.08	0.1	0.4	0.24	0.16
si	1	0.5	1.25	1.5	1.25	0.2	2	0.8	2	1.2
	0.000948167	0.00095	0.001132	0.0012	0.00166	0.00189	0.00213	0.00379	0.00647	0.00742
	(10.6.2.4.1.8.3.5.7.9)									
$S_{sotmala}$ =	93662.95	=	**93663**							

	10	6	2	4	1	8	3	5	7	9
hiri	0.15	0.08	0.12	0.04	0.22	0.04	0.06	0.4	0.42	0.32
si	1	0.5	1.25	1.5	1.25	0.2	2	0.8	2	1.2
riti	0.6	0.32	0.4	0.12	0.5	0.08	0.1	0.4	0.24	0.16
Qriti	157.2	83.84	104.8	31.44	131	20.96	26.2	104.8	62.88	41.92
TC =	**272907**									

LAMPIRAN 3. Contoh Perhitungan Penelitian Tahap II

Tahap I - D1000

I	ri	ki	hi	si	ti	rihi	riti		
1	1	80	0.22	1.25	0.5	0.22	0.500		
2	2	80	0.06	1.25	0.2	0.12	0.400		
3	2	40	0.03	2	0.05	0.06	0.100		
4	4	40	0.01	1.5	0.03	0.04	0.120		
5	4	20	0.1	0.8	0.1	0.4	0.400		
6	4	20	0.02	0.5	0.08	0.08	0.320		
7	6	10	0.07	2	0.04	0.42	0.240		
8	8	10	0.005	0.2	0.01	0.04	0.080		
9	8	10	0.04	1.2	0.02	0.32	0.160		
10	10	10	0.015	1	0.06	0.15	0.600		
		320		11.7		1.85	2.92	x =	0.1

N =	t =	TPC =	E =	D =	H =	π	$\hat{\pi}$	i	θ
10	10	1	250	1000	3	1	5	0.15	200
A =	B =	C =	K =					a	b
12.92	1.85	11.7	320					10949.7	1898.24

QLB = -0.9056

QSOL = 271 271.41

Qmin = **271**

	1	2	3	4	5	6	7	8	9	10
hiri	0.22	0.12	0.06	0.04	0.4	0.08	0.42	0.04	0.32	0.15
riti	0.500	0.400	0.100	0.120	0.400	0.320	0.240	0.080	0.160	0.600
si	1.25	1.25	2	1.5	0.8	0.5	2	0.2	1.2	1
	0.00166	0.00113	0.00213	0.00121	0.00379	0.00095	0.00647	0.00189	0.00742	0.00095

(10.6.2.4.1.8.3.5.7.9)

Ssolmaks = 345584 = **301982.5**

K 77.1636 = **77**

	10	6	2	4	1	8	3	5	7	9
hiri	0.15	0.08	0.12	0.04	0.22	0.04	0.06	0.4	0.42	0.32
si	1	0.5	1.25	1.5	1.25	0.2	2	0.8	2	1.2
riti	0.6	0.32	0.4	0.12	0.5	0.08	0.1	0.4	0.24	0.16
Qriti	157.2	83.84	104.8	31.44	131	20.96	26.2	104.8	62.88	41.92

TC = **5611789.97**

Tahap II

I	ri	ki	hi	si	ti	rihi	riti		
1	1	80	0.22	1.25	0.5	0.22	0.500		
2	2	80	0.06	1.25	0.2	0.12	0.400		
3	2	40	0.03	2	0.05	0.06	0.100		
4	4	40	0.01	1.5	0.03	0.04	0.120		
5	4	20	0.1	0.8	0.1	0.4	0.400		
6	4	20	0.02	0.5	0.08	0.08	0.320		
7	6	10	0.07	2	0.04	0.42	0.240		
8	8	10	0.005	0.2	0.01	0.04	0.080		
9	8	10	0.04	1.2	0.02	0.32	0.160		
10	10	10	0.015	1	0.06	0.15	0.600		
		75		11.7		1.85	2.92	x =	0.1

A =	B =	C =	K =					a	b	
0.0392	1.85	0.117	97					10949.68	1898.244	
Q =	**492**	491.2349	**0.765088**							
	10	6	2	4	1	8	3	5	7	9
hiri	0.15	0.08	0.12	0.04	0.22	0.04	0.06	0.4	0.42	0.32
riti	0.006	0.0032	0.004	0.0012	0.005	0.0008	0.001	0.004	0.0024	0.0016
si	0.01	0.005	0.0125	0.015	0.0125	0.002	0.02	0.008	0.02	0.012
	0.073171	0.073193	0.087432	0.094563	0.128467	0.145985	0.166667	0.292398	0.502392	0.57554
	(10.6.2.4.1.8.3.5.7.9)									
Ssolmaks =	3234.298	=	**3234.298**							
K	140.0904	=	**140**							
	10	6	2	4	1	8	3	5	7	9
hiri	0.15	0.08	0.12	0.04	0.22	0.04	0.06	0.4	0.42	0.32
si	0.01	0.005	0.0125	0.015	0.0125	0.002	0.02	0.008	0.02	0.012
riti	0.006	0.0032	0.004	0.0012	0.005	0.0008	0.001	0.004	0.0024	0.0016
Qriti	2.04	1.088	1.36	0.408	1.7	0.272	0.34	1.36	0.816	0.544
TC =	**8976.432**	339.8645								

tahap III

I	ri	ki	hi	si	ti	rihi	riti		
1	1	80	0.22	0.0125	0.005	0.22	0.005		
2	2	80	0.06	0.0125	0.002	0.12	0.004		
3	2	40	0.03	0.02	0.0005	0.06	0.001		
4	4	40	0.01	0.015	0.0003	0.04	0.001		
5	4	20	0.1	0.008	0.001	0.4	0.004		
6	4	20	0.02	0.005	0.0008	0.08	0.003		
7	6	10	0.07	0.02	0.0004	0.42	0.002		
8	8	10	0.005	0.002	0.0001	0.04	0.001		
9	8	10	0.04	0.012	0.0002	0.32	0.002		
10	10	10	0.015	0.01	0.0006	0.15	0.006		
		140		0.117		1.85	0.0292	x =	0.1

N =	t =	TPC =	E =	D =	H =			l	teta	
10	0.01	1	250	1000	1	1	5	0.15	200	
A =	B =	C =	K =					a	b	
0.0392	1.85	0.117	97					10949.68	1898.244	
Q =	**492**	491.2349								
	10	6	2	4	1	8	3	5	7	9
hiri	0.15	0.08	0.12	0.04	0.22	0.04	0.06	0.4	0.42	0.32
riti	0.006	0.0032	0.004	0.0012	0.005	0.0008	0.001	0.004	0.0024	0.0016
si	0.01	0.005	0.0125	0.015	0.0125	0.002	0.02	0.008	0.02	0.012
	0.073171	0.073193	0.087432	0.094563	0.128467	0.145985	0.166667	0.292398	0.502392	0.57554
	(10.6.2.4.1.8.3.5.7.9)									
Ssolmaks =	3234.298	=	**3234.298**							
K	140.0904	=	**140**							
	10	6	2	4	1	8	3	5	7	9
hiri	0.15	0.08	0.12	0.04	0.22	0.04	0.06	0.4	0.42	0.32
si	0.01	0.005	0.0125	0.015	0.0125	0.002	0.02	0.008	0.02	0.012
riti	0.006	0.0032	0.004	0.0012	0.005	0.0008	0.001	0.004	0.0024	0.0016
Qriti	2.04	1.088	1.36	0.408	1.7	0.272	0.34	1.36	0.816	0.544
TC =	**8976.432**									

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Theory and Methodology

Economic scheduling of products with N components on a single machine

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Abstract

We consider a single machine scheduling problem where components of N different types are made and then assembled to final products of a single type. The fixed setup time and the setup cost occur whenever the machine is switched from production of one type to another. The demand for the final products is assumed to be constant and the planning time horizon is infinite. We propose a heuristic batching and sequencing algorithm which is based on a repetitive schedule to minimize the total production cost. The total production cost consists of work-in-process inventory costs of components, setup costs, and the inventory holding cost plus the cost of completing orders for the final products. We demonstrate that our algorithm is superior to the well known lot sizing (Economic Order Quantity) rule for the problem. An example is also given to illustrate the solution procedure.

Keywords: Scheduling, Components, Heuristics

1. Introduction

This paper considers the problem of scheduling products on a single machine where products are composed of N different components. A single machine makes the components of each type, but it incurs a fixed delay and changeover cost whenever it changes from one type to another. If components are sent to the assembly shop in incomplete kits for final assemblies, those delivered components will not be useful until the remaining components are added to make those kits complete. This type of production is quite common in manufacturing industries. For example, a hobbing machine produces different types of gear boxes which are assembled to power transmission gear boxes.

The literature on problems with setup and changeover costs has grown sharply in recent years. Most of this research has concerned in batch sequencing problems. Among the notable papers on batch sequencing problems are Naddef and Santos [7], Santos and Magazine [8], Baker [1], and Dobson, Karmarkar and Rummel [3]. In batch sequencing problems, the objective is to determine the sequence for a given set of n jobs so as to minimize the total flow time. Coffman, Nozari and Yannakakis [2] developed an efficient algorithm to minimize the total flow time for two subassembly scheduling problems, under the assumptions of equal setup delay for each subassembly and products consisting of one sub-

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assembly of each type. Keila [6] considered the scheduling problem in which a machine works constantly and produces two types of fluid products. He developed a closed form solution based on the long-run average discounted cost criterion with the assumption that whenever there is a positive amount of both types of products, the products are mixed immediately and shipped away.

In this paper, we consider deterministic scheduling problems on a single machine where final products of single type consist of N components. We assume that the demand for final products is constant and the planning time horizon is infinite. One of the most commonly used objectives in the previous researches is the minimization of the total flow time. If the firm has some ability to manipulate demand and due-dates, this objective is a sufficient measure. However, we should also consider the inventory holding cost of final products since the final products are manufactured during the early part of the cycle and used at a uniform rate throughout the cycle. Therefore, our choice will be that of minimizing the total production cost which consists of work-in-process inventory holding costs of components, changeover costs, and inventory holding cost plus the cost of completing orders for the final products.

If there is an external demand for final products (parent items), N component items that are produced to meet production plan requirements for parent items have dependent demand. Such parent-component dependent relationships, which are expressed in bills of materials, greatly increase the complexity of inventory management. The problem considered here deals with the determination of optimum lot sizing as well as optimum production sequencing for N components on a single machine given constant demands for final products. The problem is difficult since batching decisions and sequencing decisions are both at issue. The only difference between our problem and the well-known ELSP (Economic Lot Scheduling Problem) is that the components cannot be sold as final products in our problem, while each product can be sold as a final product in the ELSP. Elmaghraby [4] provided an extensive overview of the ELSP problem, and Hsu [5] showed that even the question of finding a feasible schedule under some restrictions for the ELSP is computationally hard (NP-complete).

We suggest a heuristic for solving the problem of scheduling N components on a single machine. Our algorithm is an iterative batching and sequencing algorithm which is based on a repetitive schedule. The paper is organized as follows. A problem statement is given in Section 2, and the proposed solution procedure is described in Section 3. An example is given to illustrate the solution procedure in Section 4. Section 5 summarizes the results of this paper and discusses extensions.

2. Problem statement

The problem is to determine the minimum cost feasible schedule for N components on a single machine. The relevant assumptions for the problem of scheduling N components are:

1) Setup time switching to each component is independent of production order.

2) Shortages are not allowed.

3) Holding costs are directly proportional to inventory level.

4) Schedules are repetitive.

We assume that the following are known and fixed:

- i = The component number ($i = 1, 2, \ldots, N$).
- t = Time to assemble N components to make one final product.
- r_i = Number of parts of component i required in a final product.
- s_i = Setup time incurred when switching to produce component i.
- k_i = Changeover cost incurred when switching to produce component i.
- t_i = Processing time for one part of component i.
- h_i = Holding cost for one part of component i per unit time.
- K = Cost of completing an order for final products.
- D = Demand rate for final products.
- H = Holding cost for one unit of final products per unit time.

Work-in-process inventory costs are dependent upon the flow times of the components. Two types of flow times can be considered on the machine: item-flow, when a part can be delivered immediately after the machining on it is complete, or batch-flow, when a part waits at the machine at the rest of the parts in batches are completed. We only consider the batch-flow time, because a part after the machining will not be useful until the remaining parts make those kits complete.

To minimize the makespan for a collection of final products, it is best to minimize changeover delays and to make all required parts of a given component before making those of other components. Because of the above advantages, we search for the schedules that make all required parts of a given component before making those of another components. Define the following decision variables for the problem of scheduling N components on a single machine:

Q = Batch size of final products.
S = $\langle S_N(1), S_N(2), \cdots, S_N(N) \rangle$: production sequence of components.

Here, $S_N(i)$ specifies the type of component from which the i-th production is selected. For example, suppose that $N = 2$ and the schedule begins with the production of component 2, followed by the production of component 1. Then the production sequence of this schedule is represented by $\langle 2, 1 \rangle$. The production sequence and the batch size will affect work-in-process inventory costs and changeover costs, because each part of the component has different setup time, and processing time.

We assume that the raw materials for each component are purchased just when they are needed. Then the batch-flow time of a component can be measured from the start time of the first part of the component on the machine to the production and assembly time of the whole batch. Therefore, each component has different start time that is determined by a production sequence of N components.

Let $s_{[i]}$ denote the setup time of the i-th component under the production sequence S and the batch size Q. Similarly, $r_{[i]}$, $t_{[i]}$, $h_{[i]}$ denote the number of parts required in a final product, the processing time, and the inventory holding cost per unit time of the i-th component under the production sequence S and the batch size Q, respectively. Then the start time, ST_i, of the first part of the i-th component on the machine under S and Q is such that

$$\sum_{k=1}^{i-1} \left(S_{[k]} + Q r_{[k]} t_{[k]} \right). \tag{1}$$

Throughout the paper, the summation from the index one to zero is defined to be zero. Also the following notations are used to facilitate the problem formulation:

$$A = t + \sum_{i=1}^{N} r_i t_i,$$

$$B = \sum_{i=1}^{N} h_i r_i,$$

$$C = \sum_{i=1}^{N} s_i,$$

$$E = \sum_{i=1}^{N} k_i. \tag{2}$$

The completion time of the whole batch given the batch size Q corresponds to the production and assembly time of all components. Therefore, the completion time, T, is given by

$$T = Q\left(t + \sum_{i=1}^{N} r_i t_i \right) + \sum_{i=1}^{N} s_i$$
$$= AQ + C. \tag{3}$$

Then the production rate, P, of the final products is a function of Q, and is such that

$$P = \frac{Q}{T} = \frac{Q}{AQ + C}. \tag{4}$$

Since the completion time of the whole batch is T, the batch-flow time of the i-th component is given by

$$FT_i = T - ST_i = Qt + \sum_{k=i}^{N} \left(S_{[k]} + Q r_{[k]} t_{[k]} \right). \tag{5}$$

Then the average work-in-process inventory cost per unit time, $\overline{\mathrm{WIP}}(Q, S)$, given the batch size Q and the production sequence S, can be expressed as

$$\mathrm{WIP}(Q, S)$$
$$= \frac{D}{Q} * \sum_{i=1}^{N} \left(Q r_{[i]} h_{[i]} \right) * \mathrm{FT}_i$$
$$= QBDt + D \sum_{i=1}^{N} \sum_{k=i}^{N} h_{[i]} r_{[i]} \left(S_{[k]} + Q r_{[k]} t_{[k]} \right). \tag{6}$$

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Let TSC denote the sum of changeover costs per unit time for a given batch of size Q. Then TSC is such that

$$\mathrm{TSC} = \frac{D}{Q} \cdot \sum_{i=1}^{N} s_i = \frac{DE}{Q}. \tag{7}$$

The final products are assumed to be manufactured at a uniform rate (P) during the early part (T) of the inventory cycle and used at a uniform rate (D) throughout the cycle. Let FIC denote the inventory holding cost of final products per unit time. Then this model is known as the economic production quantity (EPQ), and it is well known that FIC is given by

$$\mathrm{FIC} = \frac{HQ}{2}\left(1 - \frac{D}{P}\right) + \frac{DK}{Q}$$
$$= \frac{HQ}{2}(1 - AD) + \frac{DK}{Q} - \frac{CDH}{2}. \tag{8}$$

Then the total production cost per unit time, $\mathrm{TC}(Q, S)$, given the batch size Q and the production sequence S, corresponds to

$$\mathrm{TC}(Q, S) = \mathrm{WIP}(Q, S) + \mathrm{TSC} + \mathrm{FIC}. \tag{9}$$

As a constraint, the production rate of the final products should be greater than or equal to the demand rate. Therefore, we can obtain the following lower bound, QLB, on the batch size:

$$\mathrm{QLB} = CD/(1 - AD). \tag{10}$$

The problem of scheduling products with N components is, then, to find the optimum batch size and the optimum production sequence of components that minimizes $\mathrm{TC}(Q, S)$ of (9) under the constraint on the batch size of (10).

3. Solution procedure

The joint determination of the optimum batch size and the optimum production sequence of components requires a total evaluation for all possible combinations of Q and S. However, if Q is fixed, there exists a simple structure which allows an optimal production sequence of components by the following proposition:

Proposition 1. *Suppose that Q is fixed. production sequence which minimizes TC((9) can be obtained by renumbering the N nents so that*

$$\frac{h_{[1]} r_{[1]}}{s_{[1]} + Q r_{[1]} t_{[1]}} \le \frac{h_{[2]} r_{[2]}}{s_{[2]} + Q r_{[2]} t_{[2]}} \le \cdots \le \frac{h_{[N]} r_{[N]}}{s_{[N]} + Q r_{[N]} t_{[N]}}.$$

Proof. See the Appendix.

Rule (11) indicates that in order to determine production sequence for N components using in values, the batch size Q should be known.

If the production sequence, S, is fixed, th $\mathrm{TC}(Q, S)$ of (9) is only a function of Q, and t optimum batch size given the production sequen can be easily obtained by the following propositio

Proposition 2. *Suppose that the production sequence S is fixed, and let Q_S^* denote the optimum batch size given the production sequence S. Then Q_S^* can be obtained by*

$$Q_S^* = [2D(E + K)]^{1/2} \times \left\{ H(1 - AD) + 2DIB + 2D \sum_{i=1}^{N} h_{[i]} r_{[i]} \sum_{k=i}^{N} r_{[k]} t_{[k]} \right\}^{-1/2}. \tag{12}$$

Proof. See the Appendix.

If one decision variable (Q or S) is given, the other decision variable can be easily determined. Therefore, an iterative solution procedure is appropriate. Also if the possible range of the optimum batch size can be found, the iterative solution procedure will be more effective.

Suppose that all components are available at time zero. In this case, the batch-flow times for all components are T and the batch size is the unique decision variable. This optimum batch size can be obtained by the following lemma:

Lemma 1. *Suppose that all components are available at time zero. If there does not exist any constraint on the batch size, the optimum batch size, QSOL, is such that*

$$\mathrm{QSOL} = \sqrt{\frac{2D(E+K)}{2ABD + H - ADH}}. \tag{13}$$

Proof. See the Appendix.

Utilizing the result of Lemma 1, we can reduce the interval on the optimum batch size by the following proposition:

Proposition 3. *Let* QMIN *denote the maximum of* QSOL *and* QLB *of* (10). *Then the optimum batch size,* Q^*, *which minimizes* TC(Q, S) *of* (9), *is greater than or equal to* QMIN.

Proof. See the Appendix.

The solution procedure begins with a determination of the initial value of Q. Since the optimum batch size is greater than or equal to QMIN, we select QMIN as the initial value of Q. Then the initial production sequence of components is obtained by the index rule of (11) given the batch size QMIN. The procedure improves the initial values by iteration as outlined below:

Step 1. Calculate QMIN and let QMIN be the initial value of Q.

Step 2. Determine the production sequence, S, of components by the index rule of (11). Then calculate the total production cost, TC(Q, S), of (9). If the difference between the current solution and the previous solution is less than the predetermined tolerance limit, stop the procedure. Otherwise, go to Step 3.

Step 3. Calculate the optimum batch size, Q_S^*, of (12). If the solution is unchanged, stop the procedure. Otherwise, substitute Q_S^* for a new value of Q and go to Step 2.

One of the commonly used lot-sizing rule for the MRP (Material Requirement Planning) system is the fixed order quantity (FOQ). The FOQ is a typical static lot sizing rule under which the lot size is a predetermined fixed quantity. The lot size could be predetermined by the economic order quantity (EOQ) formula. Therefore, one of the traditional methods of handling our problem is to treat the components independently of the final products (this will be referred to as ID solution). In ID solution, the batch size of final products is determined by the EOQ, and then the production sequence of N components is obtained by the index rule of (11). It is well known that the EOQ is given by

$$\mathrm{EOQ} = \sqrt{2DK/H}. \tag{14}$$

The ID solution can be used as an initial solution of the algorithm, and it may enhance the performance

Table 1
Input data for components

i	b_i E_i	t_i (·)	h_i (\$/unit/day)	S_i (day)	E_i (day)	[illegible]	[illegible]	Qriti	Qriti+Sk
1	1	80	0.22	- 0.0125	0.005	0.22	0.005	0,4	0,4325
2	2	80	0.06	- 0.0125	0.002	0.12	0.004	0,32	0,3325
3	2	40	0.03	0.02	0.0005	0.06	0.001	0,08	0,1
4	4	40	0.01	0.015	0.0003	0.04	0.0012	0,096	0,111
5	4	20	0.1	0.008	0.001	0.4	0.004	0,32	0,328
6	4	20	0.02	0.005	0.0008	0.08	0.0032	0,256	0,2561
7	6	10	0.07	0.02	0.0004	0.42	0.0024	0,192	0,212
8	8	10	0.005	0.002	0.0001	0.04	0.0008	0,064	[illegible] 0,066
9	8	10	0.04	0.012	0.0002	0.32	0.0016	0,128	0,14
0	10	10	0.015	0.01	0.0006	0.15	0.006	0,48	0,49
	49	320	0.57	0.117	0.0109	1.85	0.0282	2,912	2.[illegible]

of the solution procedure. Therefore, the worst case performance of our proposed method is equivalent to that of the ID solution. Also the maximum number of iterations for our algorithm can be limited to a reasonable number by selecting the suitable value of the predetermined tolerance limit on the total production cost. In the next section, a ten component example problem is used to illustrate the solution procedure.

4. Example problem

N = 10 komponen

Consider a ten component problem with assembly time $t = 0.01$ day/unit. The tolerance limit on the total production cost is given by \$1.0/day. The input data for the final products are as follows: $K =$ \$250/order, $D = 20$ units/day, $H =$ \$3/day/unit. The input data for N components are given in Table 1.

Using the input data for final products and components, we can obtain the following values:

$A = t + \sum_{i=1}^{N} r_i t_i = 0.0392$,

$B = \sum_{i=1}^{N} h_i r_i = 1.85, C = \sum_{i=1}^{N} S_i = 0.117$,

$E = \sum_{i=1}^{N} k_i = 320$,

$\mathrm{QLB} = CD/(1 - AD) = 11$.

According to (9), the total production cost, $TC(Q, S)$, given the batch size Q and the production sequence S, for the example problem is expressed as

$$TC(Q, S) = 11{,}400/Q + 0.694Q - 3.51 + 20\sum_{i=1}^{N} h_{[i]} r_{[i]} \sum_{k=i}^{N} \left(Q r_{[k]} t_{[k]} + S_{[k]}\right).$$

Iteration 1. In Step 1,

$$\mathrm{QSOL} = \sqrt{\frac{2D(E+K)}{2ABD - ADH + H}} = 80.$$

Then QMIN = maximum {QSOL, QLB} = 80 is selected as the initial value of Q. In Step 2, we obtain

$S = \langle 10, 6, 4, 2, 1, 3, 8, 5, 7, 9 \rangle$

using the index rule of (11). Then

$TC(Q, S) =$ \$226.6/day.

In Step 3, we obtain $Q_S^* = 103$ using Eq. (12), and go to Step 2.

Iteration 2. In Step 2, we obtain

$S = \langle 10, 6, 2, 4, 1, 8, 3, 5, 7, 9 \rangle$,

with $Q = 103$. Then

$TC(Q, S) =$ \$219.4/day.

Since the difference between this cost and the previous total production cost is greater than the predetermined tolerance limit, go to Step 3. In Step 3, we obtain $Q_S^* = 103$. Since the solution is unchanged, stop the procedure.

The performance of our proposed method can be compared with that of the ID solution. For the example problem, the ID solution produces the total production cost of \$257.1/day, compared with \$219.4/day by our method. The corresponding Q and S for the ID solution are as follows:

$Q = \sqrt{2DK/H} = 58$,

$S = \langle 10, 6, 4, 2, 1, 3, 8, 5, 7, 9 \rangle$.

5. Conclusions

This paper considers a single machine scheduling problem where components of N different types are made and then assembled to final products of single type. The problem is to determine the optimum batch size and the optimum production sequence of components within each batch so as to minimize the total production cost. The formulation studied here is a static single machine scheduling problem, for which a simple solution procedure is developed.

We assumed that the production schedule is repetitive. However, the optimal schedule may be non-repetitive. Hence, further research is necessary to address non-repetitive schedules for scheduling N component problems. We also assume that the final products are delivered in batches. In a certain situation, it may be economical to deliver individual final product at the time of assembly rather than to deliver the final products in batches. If the final products are delivered individually, the problem appears to require greater effort.

We also leave the obvious generalization to the problem of scheduling N components when the demand for the final products is stochastic. The general approach for the problem of scheduling N components will require greater effort, and we hope the results obtained in this paper will stimulate further research in this area.

Appendix

Proof of Proposition 1. Suppose that S is a schedule such that (11) does not hold. Then there exist two components i and j adjacent in S such that

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$$\frac{h_i r_i}{s_i + Q r_i t_i} > \frac{h_j r_j}{s_j + Q r_j t_j}. \tag{A.1}$$

Let S' be the schedule formed by interchanging two components i and j (see Fig. A.1). Then the start time, T_a, of component i under S corresponds to the start time of component j under S'. Similarly, the finish time, T_b, of component j under S corresponds to that of component i under S'. Therefore, the work-in-process inventory costs for any other components do not change.

Let WIP(S) denote the sum of work-in-process inventory cost for components i and j under S. Then the flow times of component i and component j under S are $T - T_a$ and $T - T_a - Q r_i t_i - s_i$, respectively. Therefore, WIP(S) is expressed as follows:

$$\begin{aligned}\mathrm{WIP}(S) &= Q h_i r_i (T - T_a) \\ &\quad + Q h_j r_j (T - T_a - Q r_i t_i - s_i). \end{aligned} \tag{A.2}$$

Similarly, the sum of work-in-process inventory cost, WIP(S'), for components i and j under S', corresponds to

$$\begin{aligned}\mathrm{WIP}(S') &= Q h_j r_j (T - T_a) \\ &\quad + Q h_i r_i (T - T_a - Q r_j t_j - s_j). \end{aligned} \tag{A.3}$$

Utilizing (A.1)–(A.3), we can obtain the following result:

$$\begin{aligned}&\mathrm{WIP}(S) - \mathrm{WIP}(S') \\ &\quad = Q h_i r_i (Q r_j t_j + s_j) - Q h_j r_j (Q r_i t_i + s_i) > 0. \end{aligned} \tag{A.4}$$

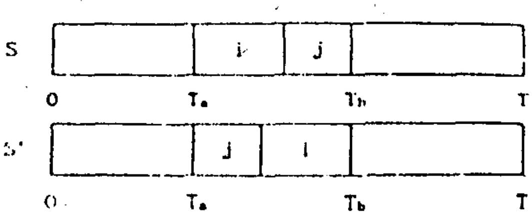

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Fig. A.1. Gantt diagram of the schedules S and S'.

Therefore, the interchange of the two adjacent components i and j brings a net decrease in the total production cost and the proof is now complete. □

Proof of Proposition 2. Utilizing (6), (7), and (8), TC(Q, S) of (9) can be expressed as

$$\begin{aligned}\mathrm{TC}(Q, S) &= QD\left(tB + \sum_{i=1}^{N} h_{[i]} r_{[i]} \sum_{k=i}^{N} r_{[k]} t_{[k]} \right) \\ &\quad + D \sum_{i=1}^{N} h_{[i]} r_{[i]} \sum_{k=i}^{N} s_{[k]} + \frac{D(E+K)}{Q} \\ &\quad + \frac{HQ(1-AD)}{2} - \frac{CDH}{2}. \end{aligned} \tag{A.5}$$

If the production sequence S is fixed, TC(Q, S) of (A.5) is only a function of Q, and it is easy to show that TC(Q, S) is convex in Q. Let Q_S^* denote the optimum batch size given S. Then Q_S^* which minimizes TC(Q, S) of (A.5) is obtained by

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$$\begin{aligned}Q_S^* &= [2D(E+K)]^{1/2} \\ &\quad \times \left\{ H(1-AD) + 2DtB \right. \\ &\quad \left. + 2D \sum_{i=1}^{N} h_{[i]} r_{[i]} \sum_{k=i}^{N} r_{[k]} t_{[k]} \right\}^{-1/2}. \end{aligned}$$

Proof of Lemma 1. Suppose that all components are available at time zero. Then the batch flow times for all components are T. Therefore, the total work-in-process inventory cost, WIP(Q, S), per unit time given Q and S, is such that

$$\begin{aligned}\mathrm{WIP}(Q, S) &= \frac{D}{Q} * T * \sum_{i=1}^{N} Q h_i r_i \\ &= ABDQ + BCD. \end{aligned} \tag{A.6}$$

Utilizing (7), (8), and (A.6), we can show that TC(Q, S) is expressed as

$$\begin{aligned}\mathrm{TC}(Q, S) &= Q\left[ABD + \tfrac{1}{2} H(1-AD) \right] \\ &\quad + \frac{D(E+K)}{Q} + BCD - \frac{CDH}{2}. \end{aligned} \tag{A.7}$$

Note that $TC(Q, S)$ is convex in Q. If we relax the constraints on the batch size, the optimal batch size, QSOL, is obtained by

$$\mathrm{QSOL} = \sqrt{\frac{2D(E+K)}{2ABD + H - ADH}}.$$

Proof of Proposition 3. If we compare Q_S^* of (12) and QSOL of (13), we can find the following relationship:

$$\begin{aligned} AB &= tB + \sum_{i=1}^{N} r_i t_i \sum_{i=1}^{N} h_i r_i \\ &\geq tB + \sum_{i=1}^{N} h_{[i]} r_{[i]} \sum_{k=1}^{N} r_{[k]} t_{[k]}. \end{aligned} \tag{A.8}$$

Therefore, we can obtain the following result:

$$Q_S^* \geq \mathrm{QSOL} \quad \text{for any production sequence } S. \tag{A.9}$$

Let QREL denote the optimum batch size which minimizes $TC(Q, S)$ of (9) with relaxing the constraint on the batch size of (10). Then QREL corresponds to one of Q_S^*, and we derive the following relationship:

$$\mathrm{QREL} \geq \mathrm{QSOL}. \tag{A.10}$$

Now we consider the constraint on the batch size. Then the optimum batch size Q^* which minimizes $TC(Q, S)$ of (9) corresponds to

$$Q^* = \begin{cases} \mathrm{QLB} & \text{if } \mathrm{QREL} < \mathrm{QLB}, \\ \mathrm{QREL} & \text{otherwise}. \end{cases} \tag{A.11}$$

Therefore, we can derive the following result:

$$\begin{aligned} Q^* &= \mathrm{maximum}\{\mathrm{QLB}, \mathrm{QREL}\} \\ &\geq \mathrm{maximum}\{\mathrm{QLB}, \mathrm{QSOL}\} \\ &= \mathrm{QMIN}. \end{aligned}$$

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INVESTING IN REDUCED SETUPS IN THE EOQ MODEL*

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This paper . . . the observation that the Japanese have devoted much time and energy to decreasing setup costs in their manufacturing processes and that there has been little in the way of a formal framework available to use to think about such efforts. The object of this paper is to begin to provide such a framework. The framework developed identifies only one aspect of the advantages of reducing setups, namely reduced inventory related operating costs. The other advantages, such as improved quality control, flexibility, and increased effective capacity, are not accounted for in this paper. Nevertheless, substantial reductions in setups may be warranted based solely on the benefits identified in this paper. The approach taken here introduces an investment cost associated with changing the (current) setup level and adds a per unit time amortization of this cost to the other costs identified in the standard EOQ model. The general problem becomes that of minimizing the sum of a convex and a concave function. In two special cases, the minimization can be carried out explicitly. In one of these cases, numerous interpretations of the results are made, including comparisons of Japanese and American practices. For example, holding other parameters constant, there is a critical sales level such that investment is made in reducing setups if and only if the sales rate is above that level. When such investment is made, the optimal lot size is independent of the sales rate. The paper also addresses the joint selection of the setup cost and the sales rate. Selection of the sales rate is seen as incorporating explicit production and holding costs into the classical monopolist's pricing problem. An explicit solution is obtained for the model postulated.

(PRODUCTION/INVENTORY; EOQ; OPERATING CHARACTERISTICS)

JIT: butuh waktu banyak
Pengurangan SU, yi pesan inv yg lebih dg b. operasi

1. Introduction

...ch has appeared (e.g., Lehner 1981, Wheelwright 1981, and Holusha 1983) about ...periority of the production and inventory management techniques used in Japan ...pared to those used in the United States. In particular, the Just-In-Time (JIT) ...m of inventory management has received a great deal of attention. Henderson ...) says, "In the history of the Industrial Revolution [JIT] may rank with inter...geable parts, precision gauge blocks, assembly lines, time and motion studies, and ...ered conveyors." Descriptions of the system can be found in Sugimori et al. (1977), ... (1981, 1983), Monden (1981a, b), Schonberger (1982) and Shingo (1981). Schon...ger (1982) and Hall (1983) give particularly useful discussions of the relevant issues. ...particular, Hall (1983) devotes considerable attention to the identification and ...ementation of reduced setup times and costs, and Schonberger (1982) illustrates ...tradeoffs associated with decreasing the setup cost in the classical EOQ model. One ...e objectives of this paper is to establish a framework for studying those tradeoffs. ...particular, this paper examines the tradeoff between the investment costs needed to ...ce the setup cost and the operating costs identified in the EOQ model. It does not ...mine the JIT system *per se*; JIT systems, with their very low inventory levels and ...ll order quantities, arise naturally when sufficient investment is made to reduce the ...p cost. As Schonberger (1982) and Hall (1983) make clear, the benefits of reducing ...setup cost transcend the benefits identified in the EOQ model alone. (For example, improved quality control, flexibility, and increased effective capacity tend to result.) However, it is appropriate to study the tradeoffs identified in this paper to determine the extent to which reductions in setups can be justified solely on the basis of reduced inventory related operating costs.

§2 introduces the basic model. §3 examines the general tradeoffs associated with changing the setup cost. The problem requires the minimization of the sum of a convex and a concave function. Explicit minimization of this sum can be carried out in certain ... cases, two of which are considered in subsequent sections. §4 examines the special case when the investment cost function is logarithmic. Starting from an initial setup level, each ten percent reduction costs a fixed amount. For example, it might cost $200 to reduce the setup cost to 90 from 100, and another $200 to reduce it another ten percent to 81. Numerous interpretations of the results are possible in this case. For example, holding other parameters constant, there is a critical sales level below which no investment in setup reduction is made. If the sales rate is above that level, then such investment is made and the optimal order quantity does not depend on the sales rate. §5 treats another special case, in which explicit minimization can be carried out. Here, the investment cost function is of the power form, so that each subsequent ten percent reduction in the setup costs costs a fixed percentage more than the prior such reduction.

The paper then goes on to consider the simultaneous optimization of the setup cost and the sales rate. §6 treats the optimization of the sales rate alone and §7 looks at the joint optimization. Changing the sales rate is viewed as a monopolist's pricing problem. §8 summarizes the results.

2. The Basic Model and Notation

The basic model is the classical undiscounted EOQ model, with m the deterministic, continuous sales (demand) rate, K the setup cost, c the unit purchase (production) cost, i the fractional per unit time opportunity cost of capital, and h the nonfinancial per unit per unit time holding cost. Unless indicated otherwise, each of these parameters is assumed to be strictly positive. Let $\hat{h} = ci + h$ denote the (effective) per unit per unit time holding cost. The economic order quantity is therefore $Q^* = \sqrt{2mK/\hat{h}}$, and the resulting (induced) minimal cost per unit time is $\sqrt{2mK\hat{h}}$. In this paper, the option of investing in reducing the setup cost will be available. In particular, $a_K(K)$ denotes the cost of changing the setup cost to the level K. Because the model deals with undiscounted costs, an opportunity cost of $ia_K(K)$ per unit time is therefore charged. It would be more accurate to have a discounted cost model. However, Hadley and Whitin (1963, p. 23) argue that for EOQ models the use of discounting "will give either identical results or almost identical results." They lead the reader through an exercise [2–69] that shows that the classical EOQ formula above is valid to a first order approximation, by using a Taylor expansion, in the discounted cost model. The purpose of this paper is to introduce the tradeoffs between investment in reduced setups and inventory related operating costs in as simple a framework as possible and to reveal a number of results that can be obtained within that framework. Dealing with the classical undiscounted cost EOQ model is the natural first step. The formulation of the discounted cost model and the less specific results that can be derived with it can be found in Porteus (1984).

m: deterministik
K: su cost
c: unit purchase
i: fractional per ...

Another interpretation for the cost per unit time assessed for achieving a particular setup cost level is that there is a revocable lease that specifies a fee to be paid per unit time to maintain that setup cost level. That is, one can either think of there being a

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occasion and a new setup cost level selected. In the model postulated, there is no difference between these interpretations, since the model has stationary, deterministic demand: There will be a single optimal setup cost level that should be maintained forever. However, there will be a substantial difference between these interpretations in a nonstationary, probabilistic demand environment.

3. The Optimal Setup Cost

This section considers the setup cost K to be a decision variable and seeks to minimize the sum of the (amortized) investment cost of changing K and the inventory related costs, by optimizing over both K and Q. The objective is to minimize $ia_K(K) + f(Q,K)$, where $f(Q,K)$ is the cost per unit time. Clearly the latter cost is a function of the order quantity Q and all the parameters, not just K, but it is convenient to use this notation when only the setup cost is to be varied. We assume that a_K is convex and strictly decreasing on its domain, which is assumed to be an interval. One reasonable way to approach the optimization problem is to fix K, optimize over Q to obtain $Q^*(K)$, and then optimize over K. In this case, we seek to minimize $w(K) := ia_K(K) + f_K(K)$, where $f_K(K) := f(Q^*(K), K) = \sqrt{2mK\bar{h}}$, which is strictly concave and increasing in K. Thus, minimizing $w(K)$ requires minimizing the sum of a convex and a concave function. Let K^* denote the optimal setup cost, and $Z^* = w(K^*)$. This unorthodox optimization problem actually has a great deal of structure and generality. Some results for the general case are given in Porteus (1984). Some specific observations are worth including here. First, marginal analysis does not necessarily apply. That is, making an investment in a small decrease in the setup and finding that the resulting savings do not exceed the investment cost does not necessarily mean that no reductions are warranted. For example, if K is to be optimized over only a specified interval, and a_K is linear in that interval, then the objective function is concave and the optimal setup cost will be one of the extreme points of that interval. In particular, it is possible that it would increase costs to make a small decrease in the setup (compared to a current value) but that a large savings can be achieved by taking the setup all the way down to its lowest possible level. If a_K is concave in general, then an extreme point solution also results, but it seems unlikely that a_K would be concave over the entire interval of consideration, in practice. Along this line, if a_K is approximated by a piecewise linear function, say $\bar{a}_K$, which equals a_K at each of the breakpoints, then minimizing $\bar{a}_K + f_K$ can be accomplished by evaluating the function only at the breakpoints. The result is obviously an upper bound on Z^*. In Porteus (1984), an algorithm, called the *nested breakpoint algorithm*, is developed, based on this idea and on a lower bound that comes from the monotonicity of the derivatives of convex and concave functions. Such an algorithm need not always be used, since in some cases a_K is sufficiently convex that $w(K)$ can be explicitly minimized over the interval of concern. Indeed, the emphasis of this paper is on two special cases in which such a minimization can be carried out. These special cases are considered in the following two sections.

4. The Logarithmic Cost Function Case

We have seen that in the special case in which a_K is linear over an interval, the optimal setup cost will be at one of the ends of that interval. Another more interesting special case is the logarithmic case when

$$a_K(K) = a - b\ln(K) \quad \text{for } 0 < K \le K_0,$$

where a, b and K_0 are given positive constants. K_0 can be interpreted as the original setup cost and the issue is whether or not it should be reduced and, if so, by how much. In this case, it costs a fixed amount to reduce the setup cost by a fixed percentage. For example, if K_0 is 100, it may cost \$200 to reduce the setup cost by ten percent to 90, another \$200 to reduce it to 81, and so on. Indeed, it is convenient (and general) to discuss this case in terms of the cost, θ, of making that ten percent reduction, assuming that $a_K(K_0) = 0$. ($b = -\theta/\ln(0.9)$ and $a = b\ln(K_0)$.) Note that b is proportional to θ: indeed, $b = 9.5\theta$. A little calculation shows that b is the cost of making about a 63% reduction in the setup cost.

Theorem 1. *If θ is strictly positive, then the following hold:*

(a) $a_K + f_K$ is strictly convex-concave on $[0, K_0]$. That is, there exists a nonnegative real number r in $[0, K_0]$ such that the function is strictly convex on $[0, r]$ and strictly concave on $[r, K_0]$.

(b) $a_K + f_K$ has a unique local minimum.

(c) The optimal setup cost is given by

$$K^* = \min\left(K_0, \frac{2b^2i^2}{m(ic+h)}\right).$$

(d) The resulting optimal order quantity is given by

$$Q^{**} = Q^*(K^*) = \min\left[\sqrt{\frac{2mK_0}{\bar{h}}}, \frac{2bi}{\bar{h}}\right].$$

(e) The resulting optimal holding cost per unit time and optimal setup cost per unit time both equal $\min(\sqrt{mK_0\bar{h}/2}, bi)$.

(f) The resulting optimal average inventory investment level is

$$\frac{Q^{**}c}{2} = \min\left[c\sqrt{\frac{mK_0}{2\bar{h}}}, \frac{bic}{\bar{h}}\right].$$

(g) The resulting optimal total (amortized investment plus holding and setup) cost per unit time is

$$w(K^*) = \min\left(\sqrt{2mK_0\bar{h}}, bi\left[2 - \ln\left(\frac{K_0m\bar{h}}{2b^2i^2}\right)\right]\right).$$

All proofs of theorems are given in the appendix. Part (a) warns that marginal analysis can still be misleading, but (b) guarantees that it will work. That is, suppose that, starting at K_0, a small decrease in the setup is carried out. If that experiment yields no net savings, then no further reductions should be attempted. However, if it yields positive net savings, then further reductions should be attempted and an identical absolute reduction in the setup can possibly yield a higher net savings than the first reduction. Such observations are relevant when the firm knows that the form of its cost function is as given here but does not know the specific parameters. Of course, if the parameters are known, then part (c) prescribes exactly what the setup should be.

Numerical Example

Suppose $c = 50$, $i = 0.15$, $h = 0.5$, $m = 1000$, $K_0 = 100$, and $\theta = 200$. Thus, as in the example at the beginning of this section, it will cost \$200 for each reduction of the setup cost by 10%. When $K = 100$, the resulting optimal order quantity is 157 and the total cost (per unit time) is about \$1265. The optimal setup cost is about 20, which yields an optimal order quantity of 71 and a total cost of \$1024. This reduction in cost of \$241 amounts to nearly 20%! The reduction of the setup cost from 100 to 20 requires over 15 ten percent reductions, for an investment of about \$3050, which amounts to \$458 per year. The resulting return amounts to \$695 per year, half of which is due to lower average inventory levels and half is due to lower setup costs. That return can be viewed as being about a 23% internal rate of return.

Sensitivity Analyses

What follows are interpretations of the optimal setup and lot size when a particular parameter is changed, holding the others constant. Technically, we must either think of (1) two different firms, each identical except for their different values of the parameter being discussed, (2) a single firm with different possible starting values of the parameter, or (3) a single firm with several different products, differing only in one parameter value. In particular, we cannot speak about a single firm whose parameter value for a given product has changed at some point in time, because a correct model of that situation would have the firm's decisions depend on the prospects for change in the parameters in the future. The deterministic uniform sales rate assumed by the current model clearly does not encompass such contingency planning. However, it is common practice for firms to use the EOQ formula in nonstationary settings and the folklore is that there are many such settings in which doing so yields good results. Thus, some readers may choose to think of the interpretations in that way, regardless of the qualifications provided here.

(a) *Sensitivity to the Sales Rate.* Part (c) shows that there will be no investment in reducing the setup until a firm's sales rate is large enough. In particular, some reduction in the setup is optimal if and only if $m > 2(bi)^2/(K_0h)$. Among low volume firms, changing the sales rate has no effect on the optimal setup level. However, among high volume firms whose sales rates are above the critical level, then doubling volume means that the setup level should be cut in two. Thus, as usual, a high volume firm should invest more to streamline its production process than a low volume firm. Indeed, because the cost of reducing setups is convex, a firm with twice the volume of a competitor will invest more than twice the amount in setup reduction spent by that competitor.

Consider a firm that anticipates a substantial increase in its sales rate. An analysis using this model with the firm's current sales rate would indicate a particular setup level. However, an analysis with the anticipated rate might indicate a much lower setup level. The firm might select the lower setup level before it might otherwise seem warranted. This is particularly true if there is a leadtime between attempts to reduce setup levels and implementation. Perhaps some Japanese firms, with their apparent obsession with driving setup levels down as far as possible, can be looked at in this way. Of course, as indicated above, a more realistic model that explicitly dealt with changes in the sales rate could address this issue directly.

Consider a firm with many different products, all alike except for their demand rates. Part (d) shows that, for low volume products, the optimal order quantity equals the traditional EOQ. If one product's sales rate is twice that of another, its lot size will be about 40% higher than the other's. However, for high volume products, the optimal order quantity is independent of the sales rate, and therefore the lot size for all such products will be the same, regardless of their sales rates. This result is consistent with Japanese just-in-time systems in which a standard container is used to convey the "order" (production) quantity, regardless of the product being made. This analogy is not exact, since a standard container will not hold the same number of all types of pieces, there is no guarantee that bi/h will be the same for all pieces, and it is not clear that the Japanese face a cost function of the logarithmic form assumed here. Nevertheless, a standard cart will hold a smaller number of large pieces compared to small pieces. A large piece will likely have a larger effective holding cost h and therefore a smaller optimal production quantity than a small piece. Thus, the use of standard containers for all pieces remains plausible. On the other hand, this result shows that the order quantity should not necessarily be driven down to one. Further reductions in the order quantity require either a lower cost of making a ten percent reduction or a lower cost of capital.

Part (e) shows that, for high volume products, holding costs are also independent of the sales rate. This result is understandable since the order quantity is independent of the sales rate. It also says that the setup costs incurred per unit time are independent of the sales rate. If one product's sales rate is twice that of another, setups occur twice as often with a constant order quantity. However, its optimal setup cost is half that of the other product, so the net setup costs incurred per unit time are the same for both products.

Part (g) shows explicitly how total costs vary as a function of the sales rate, as well as the other parameters. In particular, total cost is a strictly concave function of the sales rate, so that, as usual, there are economies of scale in this business.

Consider two "similar" firms, each identical except that one has the opportunity of reducing its setup level (with a cost function as specified here) and the other doesn't. (You might interpret the second firm as not recognizing the opportunity.) Consider the ratio of optimal costs for the first firm over that of the second. We now examine what happens to that ratio when the volume of both firms is changed. For low volume levels, the ratio is one: the optimal total costs are the same for both firms. When the volume increases above the critical level mentioned above, the first firm will invest in reducing setups and the second won't. Beyond this point, the ratio will drop below one: the first firm will have lower costs. This ratio will continue to decrease. Indeed, it can be shown that the ratio will continue to decrease, all the way to zero, as the sales rate increases. The ratio is concave-convex and the point where it turns from concave to convex is at $e^{2/3} \approx 1.95$ times the critical volume level. That is, the ratio will decrease at a faster rate as volume increases from the critical volume level to about twice that level. The rate will decrease after that, but the ratio will eventually approach zero. Thus, the cost advantage of the firm with the opportunity to invest in reduced setups increases as the sales volume increases. However, this cost advantage should not be overemphasized because it relates only to setups and holding costs, and does not include labor and materials.

(b) *Sensitivity to the Unit Cost.* Similar interpretations to those made for the sales rate can be made for the unit cost parameter c. There is a critical unit cost level such that investment in reduced setups is made if and only if the unit cost is above the critical level. Firms with high labor and materials costs have additional incentive to invest in reduced setups. The optimal lot size decreases as a function of the unit cost and therefore is not independent of it. For high volume products, differing only by their unit costs, the average inventory investment level is almost independent of the unit cost when h is small compared to ci.

(c) *Sensitivity to the Cost of Reducing Setups.* Part (d) can be interpreted as saying to invest in reducing setups if and only if

$$b < \frac{1}{i}\sqrt{\frac{K_0 mh}{2}}.$$

Using part (f) and remembering that $b \approx 9.5b$, the inequality can be rewritten as invest if and only if θ is less than $\bar{h}/(9.5ic)$ times the current average inventory investment level. In particular, if the cost of making a ten percent reduction in the setup is less than one tenth the average inventory investment level, then reduction in the setup should be carried out. The inequality yields a marginal analysis rationale that can be applied in some practical settings even if the cost of reducing setups is not of the logarithmic form specified here. If the average inventory investment in a piece is more than ten times the cost of making a ten percent reduction in the setup, then some reduction of the setup should be carried out. Keep reducing the setup level until the average inventory investment equals $9.5ic/\bar{h}$ times the cost of making another ten percent reduction in the setup.

Part (c) also shows how the optimal setup level varies as a function of the setup cost reduction parameter b. For large values of that parameter, reductions have no effect on the optimal setup level (or anything else). When b drops below the critical level, though, each time b is cut in two, the optimal setup is cut in four and the optimal order quantity and the average inventory holding costs are both cut in two. The total cost is not cut quite in two, because investment in setup reductions is increased. However, the percentage reduction in the total cost will increase each time the parameter b is successively cut in two.

As was done earlier, consider the ratio of the optimal costs of the two 'similar' firms, with the costs of the firm having setup reduction opportunities in the numerator, as usual. Clearly, as the parameter b is reduced down to zero, this ratio decreases from one down to zero.

Another way to make a comparison between two firms is to examine their average investment levels. (Assume the firms are identical except with respect to their costs of making setup reductions.) The one with the lower average inventory investment level reveals its better opportunities for setup reductions and therefore signals that it has lower total costs. The more dramatic the difference in average inventory investment levels is, the greater the cost advantage of the low setup firm. This may be a way to look at the great differences in average inventory investment in Japanese firms compared to American firms.

(d) *Sensitivity to the Cost of Capital.* Part (c) can also be interpreted as saying to invest in setup reductions if and only if the cost of capital i is below a critical level. Because $\bar{h}$ depends on i, the formula is less revealing than others and is not displayed here. As the cost of capital is decreased below the critical level, the optimal setup cost, order quantity, average inventory costs, and total costs all decrease. The average inventory investment level also decreases but possibly very slowly. In particular, if the nonfinancial holding cost h is negligible, then the average inventory investment level is independent of i in this interval. With inefficient capital markets, it is possible that two otherwise identical firms might have different costs of capital. The one with the lower cost of capital would have lower setups, lower order quantities, and lower costs. However, the average inventory investment levels might be very similar.

As was done earlier, consider the ratio of optimal costs of the two 'similar' firms, now considered as a function of i. When the cost of capital is high, the ratio is one: both firms have the same costs. As i approaches zero, the ratio approaches zero, and the ratio is concave in i in general.

5. The Power Cost Function Case

In this case, we assume that a_K has the form $a_K(K) = aK^{-b} - d$ for $0 < K \le K_0$, where a, b, d, and K_0 are given positive constants. K_0 can again be interpreted as the original setup cost. We again use a ten percent reduction in the setup cost as a nominal unit. In this case, we can specify that the first ten percent reduction costs a fixed

amount, say \$200, and every additional ten percent reduction costs a fixed percentage more than the last one. For example, if K_0 is 100, it may cost \$200 to reduce that cost to 90, \$210 (five percent more than \$200) to reduce it to 81, \$220.5 to reduce it to 72.90, and so on. Again, it is convenient (and general) to discuss this case in terms of that first ten percent reduction parameter, θ, and the compounding parameter, γ. (In the example above, $\theta = 200$ and $\gamma = 0.05$. The specifications are:

$$b = \frac{-\ln(1+\gamma)}{\ln(0.9)}, \qquad a = \frac{\theta K_0^b}{(0.9)^{-b} - 1},$$

and $d = aK_0^{-b}$.)

Theorem 2. *If θ and γ are strictly positive in this case, then*
(a) $a_K + f_K$ is strictly convex-concave on $[0, K_0]$.
(b) $a_K + f_K$ has a unique local minimum.
(c) The optimal setup cost is given by

$$K^* = \min\left(K_0, \left(\frac{2(abi)^2}{m\bar{h}}\right)^{1/(2b+1)}\right).$$

(d) The resulting optimal order quantity is given by

$$Q^{**} = Q^*(K^*) = \min\left[\sqrt{\frac{2mK_0}{\bar{h}}}, \left(\frac{2m}{\bar{h}}\right)^{b/(2b+1)}\left(\frac{2iab}{\bar{h}}\right)^{1/(2b+1)}\right].$$

(Recall that proofs are given in the appendix.) The expressions for the resulting optimal costs are more complicated in this case, compared to the logarithmic case, and are therefore not displayed. Of course, they can be easily calculated by substituting the values of the optimal setup and optimal order quantity into the basic formulas. Before discussing any interpretations, it is useful to reexamine the numerical example treated earlier. (Recall that $c = 50$, $i = 0.15$, and so on.) In this case, the parameter γ appears and is set equal to 0.05. The optimal setup is about 43 in this case, as compared to 20 in the logarithmic case. The optimal order quantity is now about 103 rather than 71. Since the first ten percent reduction in the setup cost is the same in both this and the logarithmic example, but subsequent such reductions become more costly in this example, the setup cost is reduced less here and the resulting savings are also less. Many of the same interpretations that were made in the logarithmic case also apply here. For example, there is a critical volume level such that investment in reducing the setup is done if and only if the sales rate is above that critical level. The critical level is not exactly the same as that for the logarithmic case, with the same value of θ, for the following reason. Although it costs the same in both cases to make a ten percent reduction in the setup (from the original level K_0), it costs less in the power case to make a smaller than ten percent reduction. This outcome is due to standardizing the functions at such a high level of reduction, namely ten percent. Standardizing the functions at something like a 0.01% reduction would make the cutoffs virtually identical. Doubling the volume (starting above the critical level) reduces the optimal setup, but not quite by one half, and, rather than leaving the order quantity unaffected, increases it, although not necessarily by much. For instance, using the above example, doubling the demand rate to 2000 from 1000 decreases the setup to 30 from 43 (about a 30% reduction), and increases the optimal order quantity to 122 from 103 (about a 19% increase).

6. The Optimal Sales Rate

This section lays the groundwork for simultaneously determining the setup cost and sales rate m, by examining the issue of optimally setting the sales rate alone. Formally, the sales rate m is now viewed as a decision variable and there is now a cost per unit time, $a_m(m)$, of changing that rate to m. This section deals with this question by considering the firm to be a monopolistic price setting firm. For example, the firm might face a linear demand curve $m = A - Bp$, where p is the unit price of the product. By inverting the demand curve to get $p = (A - m)/B$, including the direct production costs of cm per unit time, and recognizing that an optimal reorder quantity will be chosen for every m, the objective can be taken to be to minimize $g(m) := -(A - m)m/B + cm + f_m(m)$, where $f_m(m) = \sqrt{2mK\bar{h}}$, the induced optimal operating costs of the EOQ model after optimizing over the reorder quantity, viewed as a function of the sales rate in this case. That is, we can set

$$a_m(m) = -\frac{(A-m)m}{B} + cm.$$

This problem was formulated by Whitin (1955) but not solved explicitly, as is done below.

Theorem 3. *In the linear demand curve case, assuming that $A/B > c$, the following hold:*

(a) $g(m) = a_m(m) + f_m(m)$ *is strictly concave-convex in* m.

(b) *The optimal sales rate m^* is given by:*

$$m^* = \begin{cases} \hat{m} & \text{if } g(\hat{m}) < 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{where}$$

$$\hat{m} := \frac{4\tilde{m}}{3}\cos^2\left(\frac{\phi}{3}\right), \qquad \phi := \arccos\left(\frac{-Bk}{4(\tilde{m}/3)^{3/2}}\right).$$

$\tilde{m} := (A - Bc)/2$ is the classical monopoly rate (found by ignoring the setup and holding costs), and

$$k := \sqrt{K\bar{h}/2}.$$

(c) $m^* < \tilde{m}$.

(d) *An approximation of $\hat{m}$ is given by*

$$\hat{m} \approx \tilde{m} - \left(\frac{2Bk\tilde{m}}{4\tilde{m}^{3/2} - Bk}\right).$$

Part (a) warns that simply finding a stationary point when optimizing over m can lead to the worst rate, instead of the best one. Obtaining part (b) required solving a cubic equation that defines the stationary points of g. Part (c) shows that the optimal rate here is no larger than the classical monopoly rate, found by ignoring setup and holding costs. Since these latter costs can often be relatively small compared to the direct production costs, often the classical rate $\tilde{m}$ will be a good approximation of m^*. Part (d) simply uses a Taylor expansion around $\tilde{m}$ in the cubic equation mentioned above to approximate $\hat{m}$ and thereby m^*. For example, if $K = 100$, $c = 50$, $i = 0.15$, $h = 0.5$, $A = 5000$, and $B = 50$ (so that the price when $m = 1000$ is \$80), the optimal sales rate is 1236 (compared to the initial assumed rate of 1000), and the approximation given in (d) gives the same amount. In this example, the optimal net profit per period is about \$29,840 compared to \$28,735 initially. This result differs little from that obtained by using the classical rate of $\tilde{m} = 1250$, which yields a net profit only \$4 less per period. Incidentally, the check that $g(\hat{m}) < 0$ in part (b) cannot be dropped, seen by considering the example above with $A = 2750$ instead of 5000. In this case, $\hat{m} = 61$ and $g(\hat{m}) = 82$, so $m^* = 0$ since $g(0) = 0$.

Other demand curve forms lead to different forms of a_m, the investment cost function. They will not be enumerated here, as they are widely known. All that has been done here is to incorporate setup and holding costs into the cost function in the analysis. The next step is to combine the investment in setup reduction with this analysis.

7. The Simultaneously Optimal Sales Rate and Setup Cost

In this section, the goal is to minimize the sum of the costs of changing the sales rate and the setup cost, plus the induced optimal operating costs. To keep the analysis simple, suppose that the demand curve is linear and the cost of reducing setups is logarithmic. Thus, in this case, the objective function is

$$g(m, K) = -\frac{(A-m)m}{B} + cm + \sqrt{2mK\bar{h}} + i(a - b\ln(K)).$$

Theorem 4. *In the case of a linear demand curve, assuming that $A/B > c$, and a logarithmic cost function for setup reductions, the optimal pair of sales rate and setup cost is given by*

$$(m^*, K^*) = \begin{cases} (\bar{m}, \tilde{K}(\bar{m})) & \text{if } \tilde{m}^2 < 2ihB, \ \tilde{K}(\bar{m}) < K_0, \ \text{and } g(\bar{m}, \tilde{K}(\bar{m})) < 0, \\ (m_0, K_0) & \text{otherwise,} \end{cases}$$

where $\bar{m} := \frac{1}{2}\tilde{m} + \frac{1}{2}(\tilde{m}^2 - 2ihB)^{1/2}$ is the largest stationary point of $g(m, \tilde{K}(m))$, $\tilde{K}(m) := 2(bi)^2/m\bar{h}$, and m_0 is optimal sales rate, given in part (b) *of Theorem 3, when the setup cost is K_0.*

The proof given in the appendix shows that if any of the three conditions listed in the theorem is not satisfied, the optimal choice is (m_0, K_0). The first allows the expression for $\bar{m}$ to be evaluated, the second guarantees that the implied choice of K is feasible, and the third assures that setting $m > 0$ is optimal. It can be seen that each of the conditions can be interpreted as a lower bound on $\tilde{m}$. The first condition is redundant in the sense that if the second condition holds, then it must be computable, which implies that the first holds. For clarity, it is useful to include the first condition in the statement of the theorem. The last two conditions are independent: Using the example above, use of $A = 2900$ and $B = 50$ violates the second condition but not the third, whereas use of $A = 77000$ and $B = 1500$ violates the third condition but not the second. Explicit formulas for these lower bounds are relatively cumbersome and are not displayed here. The point is that, as in the case of optimally setting the setup alone, there is a critical "sales" level above which setup reduction is appropriate. In this case, the classical monopoly sales rate must be larger than a critical level.

8. Conclusions

This paper has developed an extension of the EOQ model in which the setup cost is viewed as a decision variable, rather than as a parameter, and the cost of selecting different values of the setup is included in the formulation explicitly. The paper also does a similar thing with the sales rate and carries out an analysis of the simultaneous selection of the sales rate and setup cost for a special case.

This paper was motivated in part by the observation that the Japanese have devoted much time and energy to decreasing setup costs in their manufacturing processes and that there has been little in the way of a formal framework available to use to think about such efforts. The object of this paper was to begin to provide such a framework. The framework developed encompasses not only altering the setup costs but the sales (demand) rate. Indeed, formally, it encompasses altering the other parameters of the EOQ model, such as the nonfinancial holding cost and the direct production cost. However, such alterations were not considered in the paper, and the emphasis of the paper is on reducing setups. Although the paper is not explicitly about Just-In-Time systems, some of the characteristics of such systems have been captured. For instance, in the case of the logarithmic cost function for reducing setups, once the sales volume is above a critical level, the optimal order quantity is independent of the sales rate. Lowered setup costs occur not only as a result of engineering effort, such as that described by Hall (1983), but also because of the introduction of robots and flexible manufacturing systems (e.g., see Ohmae 1982 and Bylinsky 1983). Thus, this paper begins to provide a framework for evaluating flexible manufacturing systems. This paper shows that substantial reductions in setups may be warranted merely on the basis of reduced inventory related operating costs. Explicit consideration of the other benefits of lowered setup costs (and times) and associated reduced lot sizes, namely improvements in quality control, flexibility, and effective capacity, will indicate even smaller setup levels than those specified here.[1]

Appendix. Proofs of Theorems

Proof of Theorem 1. Let $g_K(K) := i(a - b\ln(K)) + f_K(K)$ for all $K > 0$, which coincides with the objective function for $0 < K < K_0$. Thus,

$$g_K'(K) = -\frac{ib}{K} + \left(\frac{m\bar{h}}{2K}\right)^{1/2}, \quad \text{and} \quad g_K''(K) = \left(\frac{ib}{K^2}\right) - \left(\frac{m\bar{h}}{8K^3}\right)^{1/2}.$$

Let $r := 8\ldots$ Direct computation shows that $g_K''(K) > 0$ for $K < r$, and $g_K''(K) < 0$ for $K > r$. Thus, g_K is strictly convex-concave on $[0, \infty)$, and therefore, so is the objective function on the interval $[0, K_0]$. Hence, part (a) holds.

Parts (b) and (c) follow from the fact that $g_K'(K) = 0$ has the unique solution $K = 2(ib)^2/m\bar{h}$, which is contained in the interval in which g_K is strictly convex. The remaining parts come from substituting the optimal setup level into the appropriate formulas.

Proof of Theorem 2. The proof is similar to that of Theorem 1. In this case let

$$g_K(K) := i(aK^{-b} - d) + f_K(K).$$

for all $K > 0$. Thus,

$$g_K'(K) = -iabK^{-b-1} + \left(\frac{m\bar{h}}{2}\right)^{1/2} K^{-1/2} \quad \text{and} \quad g_K''(K) = iab(b+1)K^{-b-2} - \left(\frac{m\bar{h}}{8}\right)^{1/2} K^{-3/2}.$$

Let $r := \ldots(\ldots iab)^2/m\bar{h})^{1/(2b+1)}$. Again, direct computation shows that $g_K''(K) > 0$ for $K < r$, and $g_K''(K) < 0$ for $K > r$. Thus, as above, part (a) follows. Again, the equation $g_K'(K) = 0$ for the stationary point has the unique solution $K = (2(abi)^2/m\bar{h})^{1/(2b+1)}$, which falls within the interval in which g_K is convex. Therefore, parts (b) and (c) follow. Part (d) comes from the usual substitution.

Proof of Theorem 3. Recall the definitions of $g(m)$, $\hat{m}$, ϕ, $\bar{m}$, and k from §6. Thus,

$$g'(m) = \frac{2(m - \bar{m})}{B} + km^{-1/2} \quad \text{and} \quad g''(m) = \frac{2}{B} - \frac{km^{-3/2}}{2}.$$

Let $r := (Bk/4)^{2/3}$. Direct computation shows that $g(m)$ is strictly concave on $[0, r]$ and strictly convex on $[r, \infty)$.

Consider two cases for part (b): (1) $\bar{m} < 3(Bk/4)^{2/3}$ and (2) $\bar{m} > 3(Bk/4)^{2/3}$. We assume (1) holds and

which contradicts the assumption that $g(m) \ldots 0$, since $g(0) = 0$. By showing that $g_1(m) > 0$ for all $m > 0$, $6(Bk/4)^{2/3}m^{1/2} + Bk$. Then $Bm^{1/2}g(m) > g_1(m)$ follows directly. Now, direct computation shows that g_1 is shows that g is increasing on $[0, \infty)$, and therefore minimized at 0. Now, direct computation shows that g_1 is convex on $[0, \infty)$ and minimized explicitly at $m = (Bk/4)^{2/3}$. The result follows since g_1 evaluated at that point is 0.

Thus case (2) must hold. We seek the minimizer of g on $[r, \infty)$, since we can then minimize g by comparing the resulting objective function value to $g(0) (= 0)$. Thus, we solve $g'(m) = 0$ as $(m^{1/2})^3 - \bar{m}m^{1/2} + Bk/2 = 0$, which reveals a cubic equation in $m^{1/2}$. Standard algebra of cubic equations shows that this equation has three real roots. Examination of these roots reveals that two are positive and one is negative. Thus, the largest root yields a relative minimum and must therefore exceed r as defined above. Direct comparison reveals that this largest root is given by

$$m = \frac{4\bar{m}}{3}\cos^2\left(\frac{\phi}{3}\right),$$

where, also as in Theorem 3,

$$\phi := \arccos\left(\frac{-Bk}{4(\bar{m}/3)^{3/2}}\right).$$

If $g(\hat{m}) > 0$, then, obviously g is minimized at $m = 0$, whereas if $g(\hat{m}) < 0$, then g is minimized at $\hat{m}$. This completes the proof of part (b).

Part (c) follows from the fact that any solution of $g'(m) = 0$ must satisfy $m < \bar{m}$, as does the selection $m = 0$.

Part (d) comes from using the first order Taylor series approximation $m^{-1/2} \approx \bar{m}^{-1/2} - \frac{1}{2}\bar{m}^{-3/2}(m - \bar{m})$ in the equation $g'(m) = 0$ and solving for m.

Proof of Theorem 4. Recall the definitions of $g(m, K)$, $\bar{m}$, $\hat{K}(m)$, and m_0 in §7. Consider the three conditions appearing in the theorem: (1) $\bar{m}^2 < 2ibB$, (2) $\hat{K}(\bar{m}) < K_0$, and (3) $g(\bar{m}, \hat{K}(\bar{m})) < 0$. We show that if any of these conditions are not satisfied, then a minimizer of g is (m_0, K_0). Let

$$g_m(m) := g(m, \hat{K}(m)) = -\frac{(A - m)m}{B} + cm + 2ib\left[1 + \ln\left(\frac{k_0}{m}\right) + \frac{1}{2}\ln m\right],$$

where $k_0 := \sqrt{K_0\bar{h}/2}$.

Suppose that condition (1) does not hold. Straightforward calculation shows that $g_m'(m) > 0$ for all $m > 0$. Thus, g_m is minimized at $m = 0$. Therefore, it is optimal to not invest in reducing setups. (Assuming that it is strictly optimal to reduce setups leads to a contradiction.) Hence, an optimal choice is (m_0, K_0) as desired.

Suppose that condition (1) holds. Direct computation reveals that, in general, g_m is strictly concave-convex on $[0, \infty)$ with the critical changeover at $r := \sqrt{ibB/2}$. The equation $g_m'(m) = 0$ can be rewritten as $m^2 - \bar{m}m + ibB/2 = 0$. The largest root of that quadratic equation is given by $\bar{m}$, and it yields a relative minimum of g_m. That is, $\bar{m}$ minimizes g_m on $[r, \infty)$ and g_m is strictly increasing $[\bar{m}, \infty)$. Furthermore,

$$\min_{m \in [0, \infty)} g_m(m) = \min(0, g_m(\bar{m})).$$

Suppose that (1) holds and (2) does not. We shall show that it cannot be optimal to select $K < K_0$. Suppose, for the purpose of contradiction, that an optimal pair (m^*, K^*) satisfies $K^* < K_0$. Since $\hat{K}(m)$ is strictly decreasing in m, and $K_0 < \hat{K}(\bar{m})$, it follows that $m^* > \bar{m}$. Thus, there must exist m_1 such that $\bar{m} < m_1 < m^*$ and $\hat{K}(m_1) = K_0$. Since g_m is strictly increasing on $[\bar{m}, \infty)$, it follows that

$$g(m_1, K_0) = g_m(m_1) < g_m(m^*) = g(m^*, \hat{K}(m^*)).$$

Therefore,

$$
\begin{aligned}
g(m^*, K^*) &> g(m^*, \hat{K}(m^*)) && \text{[by Theorem 1]}\\
&> g(m_1, K_0)\\
&> g(m_0, K_0) && \text{[definition of } m_0\text{]}.
\end{aligned}
$$

Suppose that condition (1) holds, but (3) does not. Then, for feasible (m, K),

$$
\begin{aligned}
g(m, K) &> g(m, \hat{K}(m)) && \text{[by Theorem 1]}\\
&> \min(0, g(\bar{m}, \hat{K}(\bar{m}))) && \text{[above, since (1) holds]}\\
&> 0 && \text{[(3) does not hold]}\\
&= g(0, K_0)\\
&> g(m_0, K_0). && \text{[definition of } m_0\text{]}
\end{aligned}
$$

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Finally, if all three conditions hold, then for feasible (m, k),

$$g(m, k) \geq \min_k g(\bar{m}, \hat{K}(\bar{m})) \quad \text{[as above]},$$
$$= g(\bar{m}, \hat{K}(\bar{m})) \quad \text{[(3) holds]}.$$

By (2), $(\bar{m}, \hat{K}(\bar{m}))$ is feasible, and, therefore, optimal.

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ALLOCATION OF SERVERS FOR STOCHASTIC SERVICE STATIONS WITH ONE OVERFLOW STATION*

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As one of its services, the Bell System provides microwave and cable transmission equipment for the transmission of news, sports and special-event TV programs from sites that do not have permanent transmission facilities. This paper describes a model for determining quantities of this TV equipment for service stations located throughout the country. This is actually a fundamental model for allocating servers (equipment in our case) for a stochastic service system consisting of several multiserver stations with one multiserver overflow station. The flows of customers to the initial stations are independent Poisson processes, and if a customer arrives at a station when all of its servers are busy, it is either served at the overflow station or denied service, depending on whether an overflow-server is available or not. The service times are exponential random variables. There are holding costs for having the servers in the system and costs for the services, depending on where they are done. We present a myopic nonlinear programming algorithm that finds numbers of servers for the stations that serve a desired percentage of customers at near-minimum cost. As part of our analysis, we show how Hayward's and Wilkinson's approximations for the blocking probability at the overflow station extend to heterogeneous service rates.
(STOCHASTIC OPTIMIZATION; MARKOV PROCESS; OVERFLOWS FROM SERVICE SYSTEMS)

1. Introduction

This paper describes a model for selecting economical quantities of servers for a stochastic service system consisting of several multiserver stations with one multiserver overflow station. We developed this model for the purpose of finding economical inventory levels of the Bell System's part-time TV transmission equipment. The following introduction gives an overview of our study.

Since the beginning of TV broadcasting in the 1950s, the Bell System has been providing part-time local TV channels for the transmission of sports, news and special-event programs from sites that do not have permanent transmission facilities. A part-time local TV channel consists of a series of microwave or cable transmission links, usually under 25 miles, that are set up temporarily to transmit a TV program to a nearby entry into the Bell System's network, which, in turn, transmits it to designated TV stations for airing or taping. There are seven types of portable transmission equipment used for these channels: microwave transmitters and receivers; cable transmitters, repeaters and receivers; and transmit and receive diplexers. This equipment is distributed among 65 "local" pools throughout the country and four "regional" pools.

We consider these pools as four separate systems: each regional pool and the local pools it supports is modeled as a system of stochastic service stations in which the equipment are the servers. The assumptions are as follows: each type of equipment is considered separately; the requests for service occur over time at the local stations according to independent Poisson processes; requests that arrive at a local station

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